GEOMETRIC PROGRAMMING IN STRUCTURAL DESIGN

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1 INTRODUCTION

Throughout the historical development of structural analysis methods there has been a parallel, if spasmodic, interest in the techniques of structural optimisation. Until the advent of the present century this interest centred mainly on achieving uniformly stressed structures equivalent, under certain conditions, to finding a minimum weight design subject to stress constraints. One impetus for generalising the scope of these early methods occurred where emphasis was placed on the design of minimum weight aircraft structures. However, a comprehensive attack on the problem had to await both the introduction of the digital computer and the development of the finite element method.

Initial research into computerised methods centred on the structural application of optimisation techniques from the parallel field of mathematical programming[1]. Whilst this approach is appropriate for problems with few structural elements it is unsuitable for large scale problems of the type encountered in aircraft design. The reason lies in the generality of mathematical programming methods which makes them computationally expensive for complex structures with many design variables or constraints. However, these methods have the advantage that under certain conditions they can be guaranteed to converge to a local optimum.

The computational inefficiency of mathematical programming methods encouraged the development of new techniques which exploit the inherent properties of the structures problem. The essence of these new techniques is to employ algorithms based on the relatively simple mathematical form for structural optimality criteria[2,3]. These 'optimality-criteria' methods have proved particularly efficient in creating optimum designs where stiffness is critical; a situation appropriate to the design of aeronautical structures. Unfortunately, though computationally efficient, optimality-criteria methods suffer the disadvantage of lacking adequate convergence proofs. Indeed, examples can be found where such techniques have converged to non-optimal solutions.

An alternative approach is to utilise special mathematical programming methods which possess a high degree of internal mathematical structure and have potential for rapid convergence to an optimum. Such methods must also be capable of accommodating the structural optimisation problem without sacrificing the special properties which provide the initial attraction. It is from within this framework that geometric programming became recognised as one of the candidate structural optimisation methods. The standard form for the method can be applied directly to certain structural optimisation problems but modifications are required for more general structural applications. This process of adaptation has resulted in a technique which blends together the mathematical programming and the optimality-criterion methods.

The primary purpose of the present paper is, therefore, twofold: to review in section 2 the development of geometric programming as a structural optimisation technique, and subsequently to illustrate that this technique is equivalent to a transformed optimality-criterion method. The theory demonstrating this correspondence is established in section 3. The following section demonstrates that a suitable solution procedure for the most recent form of geometric programming is provided by a Newton based method with constraints treated by a projected gradient philosophy. Attention is focussed on the strong linear properties of the geometric programming method and on the availability, under certain conditions, of the Hessian matrix. A few simple examples are used to illustrate the main points of the arguments.

2 REVIEW OF GEOMETRIC PROGRAMMING

The attraction of geometric programming lies in its ability to reduce the complexity of an original or primal optimisation problem by creating a concave dual function subject to linear constraints[4]. Clearly, if the primal requires minimising the associated dual is a maximisation problem. The concavity of the dual is independent of any convexity or lack of it in the primal; similarly the dual constraints remain linear no matter how non-linear the corresponding primal problem. A further advantage occurs in certain special circumstances whereby the dual space is reduced to a single point which is the maximising point. In this case a solution to the primal problem requires a single matrix inversion.

Turning, now, to a mathematical description of the method; the primal geometric programming problem is concerned with finding a vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^t$ which solves the problem

$$\text{minimise} \quad g_0(x) = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} x_j^{a_{ij}} \quad ,$$

subject to the constraints

$$1 \geqslant g_k(x) = \sum_{i=P_{(k-1)}+1}^{P_{(k)}} c_i \prod_{j=1}^{n} x_j^{a_{ij}} \qquad k = 1 \ldots m$$

giving rise to an optimisation problem with n variables and m constraints. The following conventions and assumptions apply: $x_i \geqslant 0$ $(i = 1 \ldots n)$, $P_{(0)} = n$, $P_{(k)} - P_{(k-1)}$ is the number of terms in the kth constraint, and $c_j \geqslant 0$ $(j = 1 \ldots P_{(m)})$. Exponential terms denoted by a_{ij} are real numbers which may be positive, negative or zero. From this description of the primal optimisation problem we see that the standard method can treat only structural optimisation problems which lie entirely within the positive orthant.

As we have indicated the advantage of the method is related to the attractive properties of the associated dual formulation, which is the preferred medium for seeking a solution to the above primal problem. This dual formulation is conceived in terms of the dual problem of geometric programming which is defined by requiring a vector δ^* to maximise the product

$$V(\delta) = \left[\prod_{i=1}^{P_{(m)}} \left(\frac{C_i}{\delta_i}\right)^{\delta_i}\right] \prod_{k=1}^{m} \lambda_k(\delta)^{\lambda_k(\delta)}$$

where $$\lambda_k(\delta) = \sum_{i=P_{(k-1)}+1}^{P_{(k)}} \delta_i \qquad k = 1 \ldots m$$

subject to the linear constraints

$$\delta_i \geqslant 0 \qquad i = 1 \ldots P_{(m)}$$

$$\sum_{i=1}^{\delta} \delta_i = 1$$

and

$$\sum_{i=1}^{P_{(m)}} a_{ij}\delta_i = 0 \qquad j = 1 \ldots n$$

with the convention that $y^y = y^{-y} = 1$ for $y = 0$. Having remarked earlier on the concavity of the dual function it must be emphasised that this relates to the logarithm of the above dual function, i.e. $\ln [V(\delta)]$. This concavity, and the linearity of the constraints, gives rise to a problem which is particularly well suited to the most rapid projected-gradient solution techniques. Also, it is perfectly feasible to solve the primal problem by seeking a solution to the dual since the contrained minimum for $g_0(x)$ has the same numerical value as the constrained maximum for $V(\delta)$.

The primary limitation of standard geometric programming is the requirement that the entire primal problem be expressed in terms of polynomials with positive coefficients. A requirement which clearly limits the scope of the method in dealing with structural optimisation problems. Nevertheless, Moss and Boddy[6] have demonstrated that the standard geometric programming problem is entirely adequate for the design of certain aeronautical components. In particular, they show how the minimum weight design of integral stiffened flat panels under uniform end load (Fig.1) can be found from this technique. The design constraints for the problem being a minimum gauge requirement, a maximum stress limit and a variety of buckling constraints. The authors indicate, not only the facility for geometric programming to solve the problem, but also the property it has of illuminating the influence on the optimum of various design modifications.

Morris[7] has shown how standard geometric programming can be used to generate minimum weight designs for statically determinate pin-jointed frameworks under multiple loads and subject to stress, displacement and gauge constraints. The design variables in this case are not the usual cross-sectional areas but reciprocal areas and in this case a transformation exists whereby negative coefficients can be accommodated by the standard method. An important aspect of this paper is the establishment of a correspondence between geometric programming and the optimality-criterion method of Prager and Chern[8].

Whilst these limited applications of standard geometric programming are useful, more general problems require a variation known as complementary geometric programming. The basis of this new method is an approximation scheme which replaces any arbitrary function by a single term polynomial with a positive coefficient. Thus, for a feasible point $\tilde{x}$ a function $g(\tilde{x})$ is approximated by the term $g(x,\tilde{x})$ where

$$g(x,\tilde{x}) = g(\tilde{x}) \prod_{i=1}^{n} \left(\frac{x_i}{\tilde{x}_i}\right)^{a_i}$$

with

$$a_i = \left[\frac{x_i}{g}\frac{\partial g}{\partial x_i}\right]_{x_i=\tilde{x}_i} \qquad i = 1\ldots n$$

The essence of complementary geometric programming is to generate a sequence of standard geometric programming problems by consecutively applying the above scheme. The resulting sequence of solutions converges to a local minimum of the optimisation problem providing each approximation forms a conservative estimate to the original[9,10].

An enhanced range of problems can be solved using complementary geometric programming and one of the earliest considered is the design of a ship bulkhead. This problem, originally treated by Moe[11], is concerned with the minimum weight design of the corrugated bulkhead shown in Fig.2, subject to constraints derived from a set of ship design codes. The solution by complementary geometric programming is particularly straightforward and rapid[12]. A limited range of statically indeterminate pin-jointed structures can be solved by the method; though the procedure is somewhat clumsy[13]. The major difficulty in this later application is satisfying the subsidiary rules which ensure that the approximation procedure generates a conservative sub-problem. Nevertheless, this attempt led to a fruitful conjunction of Newton algorithms with the basic geometric programming concept.

An alternative suggestion[14] is to employ the sequential geometric programming method without the requirement that each sub-problem forms a conservative estimate to the original. Indeed, the method goes further and applies the approximation scheme to *all* constraints which are thereby reduced to single term polynomials. This new technique we call, for convenience, reduced geometric programming. The move is incisive and permits the application of geometric programming to large scale structures, typified in Fig.3, where minimum weight designs are sought, subject to stress, displacement and gauge constraints.

Reduced geometric programming does not conform to the convergence theorems applicable to complementary geometric programming[9]; but despite this lack of rigour exhibits rapid convergence. In fact, for the examples shown in Fig.3, the rate of convergence is equivalent to that of optimality-criterion methods and, as is shown in later sections, there is potential for improving this performance. In the light of these remarks it is clearly important that the method be placed on a firmer basis and in order to complete the review we attempt to satisfy this requirement in the next section. We show that the new technique is primarily a logarithmic form of the optimality-criterion method. However, unlike the standard optimality-criterion methods, geometric programming retains the objective function as part of the final formulation. The lack of explicit reference to an objective function in optimality-criterion methods of the type, similar to fully-stressed design algorithms, can lead to the convergence on non-optimal solutions. Retention of this term by the reduced geometric programming approach implies that this particular form of ill-behaviour is suppressed.

3 REDUCED GEOMETRIC PROGRAMMING

Anticipating, somewhat, the later results we consider the minimum weight design of a pin-jointed framework subject to displacement constraints, which is traditionally used to develop the optimality-criterion equations. The problem is one of finding a vector $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^t$ which:

minimises the weight of a pin-jointed structure,

$$W = \sum_{i=1}^{n} \rho \ell_i x_i$$

subject to the displacement constraints

$$\bar{u}_j \geqslant u_j = \sum_{i=1}^{n} \frac{S_i \ell_i s_{ij}}{E x_i} \quad . \qquad j = 1 \ldots m$$

This structure consists of n bars and the terms S_i, ℓ_i, x_i are respectively the tension, length and cross-sectional area of the ith bar. The tension in the ith bar by the application of a unit load at the constrained jth node is denoted by s_{ij} and Young's modulus by the constant E. It is assumed that the framework is subject to a set of externally applied single or multiple loads.

Assuming a feasible point $\tilde{x}$ and applying the approximation scheme of the previous section to the constraint equations provides the expressions:

$$\bar{u}_j \geqslant u_j = \sum_{i=1}^{n} \frac{S_i \ell_i s_{ij}}{E x_i} \simeq u_j \left(\frac{x_1}{\tilde{x}_1}\right)^{a_{1j}} \left(\frac{x_2}{\tilde{x}_2}\right)^{a_{2j}} \cdots \left(\frac{x_n}{\tilde{x}_n}\right)^{a_{nj}} \qquad j = 1\ldots m$$

with

$$a_{ij} = -\frac{\sigma_i V_i \sigma_i^{(j)}}{E u_j} \qquad \begin{array}{l} i = 1\ldots n \\ j = 1\ldots m \end{array}$$

where σ_i is the stress in bar i due to the application of the applied loads, $\sigma_i^{(j)}$ is the stress in the bar due to a unit load applied at node j and V_i is the element volume. Using these approximate constraint inequalities the associated dual geometric programming problem becomes; find a vector (δ,λ) which:

maximises
$$V(\delta) = \prod_{i=1}^{n} \left(\frac{\rho \ell_i}{\delta_i}\right)^{\delta_i} \prod_{j=1}^{m} \left(\frac{u_j}{\bar{u}_j \tilde{x}_1^{a_{1j}} \ldots \tilde{x}_n^{a_{nj}}}\right)^{\lambda_j} ,$$

subject to the normality condition

$$\sum_{i=1}^{n} \delta_i = 1 ,$$

and the orthogonality conditions

$$\delta_i - \sum_{j=1}^{m} \lambda_j \frac{\sigma_i V_i \sigma_i^{(j)}}{u_j E} = 0 \qquad i = 1\ldots n$$

where the vectors δ, λ are termed dual variables.

A clearer insight into the mathematical structure of the dual problem can be obtained by noting that suitable values for the dual variables δ_i, $i = 1\ldots n$ satisfying the normality condition are given by

$$\delta_i = \frac{\ell_i x_i}{\sum_{i=1}^{n} \ell_i x_i} = \frac{\ell_i x_i}{\mathrm{vol}} \qquad i = 1\ldots n$$

where 'vol' is the volume of the entire structure. Thus, the dual problem becomes:

maximise $$V(\delta)$$

subject to $$\frac{1}{vol} - \sum_{j=1}^{m} \lambda_j \frac{\sigma_i \sigma_i^{(j)}}{u_j E} = 0 \quad ,$$

which is the final form for the reduced geometric programming formulation as applied to the structural optimisation problem.

This establishes a particular form for the geometric programming equations and we now turn to the equations of the optimality-criterion method. The iteration formulae for the optimality-criterion method are constructed by satisfying the differential forms of the associated Lagrangian function. Normally the Lagrangian employed is associated with the displacement constrained problem given above but this cannot be used if a correspondence with geometric programming is sought. In order to satisfy this requirement we turn to a logarithmic version of the structural optimisation problem which has the same solution vector as the original, sic:

minimise $$\ln\left(\sum_{i=1}^{n} \ell_i x_i\right) \quad ,$$

subject to $$\ln(\bar{u}_j) \geqslant \ln(u_j) = \ln\left(\sum_{i=1}^{n} \frac{S_i \ell_i s_{ij}}{E x_i}\right) \quad .$$

The associated Lagrangian is written,

$$L(x,\mu) = \ln\left(\sum_{i=1}^{n} \rho \ell_i x_i\right) + \sum_{j=1}^{m} \mu_j \left\{\ln(u_j) - \ln(\bar{u}_j)\right\} \quad ,$$

with the optimality criterion given by,

$$\frac{\partial L}{\partial x_i} = \frac{\ell_i}{\sum_{i=1}^{n} \ell_i x_i} + \sum_{j=1}^{m} \frac{\mu_j}{u_j} \frac{\partial u_j}{\partial x_i} = 0$$

or

$$\frac{1}{vol} - \sum_{j=1}^{m} \frac{\mu_j}{u_j} \frac{\sigma_i \sigma_i^{(j)}}{E} = 0 \quad .$$

By setting $\mu_j = \lambda_j$, $j = 1...m$ we recover the constraint equations of reduced geometric programming. Thus, collapsing the structural optimisation problem to conform with the dual form of reduced geometric programming is equivalent to a special form of the optimality-criterion method. The correspondence is only strict at the optimum solution to the dual geometric programming problem. This restriction is not serious since the standard solution procedure normally requires locating the dual optimising point to the geometric programming problem.

The two techniques are similar in their method of operation and require a structural analysis to set up the basic optimality equations. Geometric programming uses this information to set up a sub-problem which is optimised and, therefore, accomplishes a number of iterations before requiring a further analysis step. Optimality-criterion methods by contrast perform no sub-optimisation and require an analysis at each iteration step. A further difference arises when the updating formula used to generate estimates for the optimum areas are considered. For the optimality-criterion method recourse is frequently made to heuristic arguments in order to generate suitable expressions for general problems. This expediency is not required in the case of geometric programming where a variety of linear or quadratic programming methods are available for the solution of the appropriate sub-problem.

4 SOLUTION PROCEDURE

A variety of techniques have been suggested for optimising the standard geometric programming problem which forms the kernel of the sub-problems in the reduced method[15-18]. Some of these techniques rely upon the strong linear nature of the dual formulation and employ modified simplex algorithms, others incorporate the nonlinear component of the objective function directly and use a standard nonlinear solution procedure.

In the case of reduced geometric programming the mathematical structure of the dual problem has a strong influence on the selection of a particular technique. The form of the problem implies that a limited number of the dual variables are independent and, indeed, the number of free variables is equal to the number of constraints less one. For the displacement constrained problem given in the previous section there are m constraints implying the existence of $m - 1$ variables r_i (say) $i = 1 \ldots m - 1$. Furthermore, an $(m) \times (n + m)$ transformation matrix $\{b\}$ exists such that a set of dual variables can be constructed which satisfy identically the normality and orthogonality conditions. The constraints on the dual then reduce to a simple requirement that

$$\{b\} \left\{\frac{1}{r}\right\} \geqslant 0$$

where $\left\{\frac{1}{r}\right\} = \{1, r_1, r_2, \ldots, r_{m-1}\}^t$. In addition further arguments based upon the nature of the dual problem may greatly reduce the number of these inequality constraints[13]. Reduced geometric programming also gives rise to a straightforward objective function which has first and second derivatives explicitly available in terms of the variables r_i, $i = 1 \ldots m - 1$. However, it should be noted that the mixed linear/nonlinear form for the objective $\ln [V(r)]$ gives a Hessian matrix of rank n, the number of structural elements. A direct implication is that the number of constraints must be less than or equal to the number of structural elements plus one, if a non-singular Hessian $\left\{\frac{\partial \ln V(r)}{\partial r_i \partial r_j}\right\}$ $i, j = 1 \ldots m - 1$ is to exist.

These arguments make a strong case for the application of a solution procedure based upon the projected gradient philosophy with a second order Newton ascent method. This procedure has been previously employed by Morris[13] and is described in more detail by Dinkel *et al.*[17]. These earlier papers are concerned with the most general form of geometric programming without any linear component within the objective function. In the case of reduced geometric programming provision must be made to ensure that the Hessian required by the Newton step remains non-singular. Several possibilities exist for achieving this end, including the iterative creation of an inverse Hessian. A more satisfactory procedure applies an active set strategy which reduces the number of constraints in each sub-problem to n. This has the added advantage of reducing the number of dual variables and, thus, the number of iterations required to solve each sub-problem. Having established the dual sub-problem the solution procedure comprises two stages: an ascent or Newton maximisation step, followed by a restoration step. The purpose of the restoration manoeuvre is to ensure that each iteration concludes with values of the dual variables which satisfy the reduced dual constraints.

The method is illustrated first by two small examples traditionally used to demonstrate structural optimisation algorithms, and subsequently, by a previously reported stressed constrained problem[19]. The simple examples are shown in Fig.4 and consists of a three-bar truss under multiple loads and a ten-bar cantilever subject to a single load. In both of these examples the following data apply:

modulus of elasticity 6.89×10^{7} kN/m^{2} ,
displacement limits 50.8×10^{-3}m vertical, 10.16×10^{-2}m horizontal,
gauge limits 2.54×10^{-3}m ,

with the applied loads P (Fig.4) prescribed as 444.8kN. In both cases we are seeking a minimum weight design subject to displacement and gauge constraints. The iteration history obtained by using reduced geometric programming with Newton/projected-gradient solution technique is shown in Fig.5 where, actual structural weight to minimum weight, W/W^* is plotted against the number of sub-problems solved which are termed iterations. The application of reduced geometric programming can give infeasible designs and a form of scaling is employed in the present technique to control the convergence. The results obtained by the complete method indicate the power and potential of this combination of techniques.

The third problem is concerned with the minimum weight design of a wing box type of structure subject to stress constraints. The structural model utilises 195 membrane panel and bar elements and requires 105 design variables relating to spar and panel thickness. Design details for the structure together with the results of using reduced geometric programming are shown in Fig.6 where a comparison with a fully-stressing algorithm is shown. In this instance the availability of the dual function is exploited to obtain bounds. Achieving such bounds requires a slightly different solution procedure from that described above and details may be found in Kelly *et al.*[19] and Kelly[20]. However, the correspondence between this modified technique and the optimality-criteria approach is preserved as noted by Bartholomew and Morris[21].

CONCLUSIONS

Geometric programming was conceived as a special method for solving a limited range of problems. Nevertheless, a decade of development has led to a technique which can solve a wide spectrum of problems and has given rise to a very effective procedure for structural optimisation. It has been shown that during this development the adoption of a flexible policy has created a variety of methods each suited to a particular structural application.

The final outcome of this development process was the generation of reduced geometric programming with a capacity for solving large aeronautical type structures. We have shown how this technique is equivalent to the optimality-criterion methods which can thereby be blended with the better aspects of mathematical programming. The resulting computer program is shown, in section 4, to exhibit both the rapid convergence properties of the optimality-criterion method and a basic numerical stability.

Despite the effectiveness of geometric programming it would be incorrect to assume that it should be utilised to solve all classes of structural optimisation problem. All optimisation methods are problem dependent to some degree and the selection of a particular technique is directly related to the mathematical structure encountered. The design of a complex structure may, therefore, require exploiting a group of optimisation methods and from the work presented here we conclude that geometric programming will have a place within this group.

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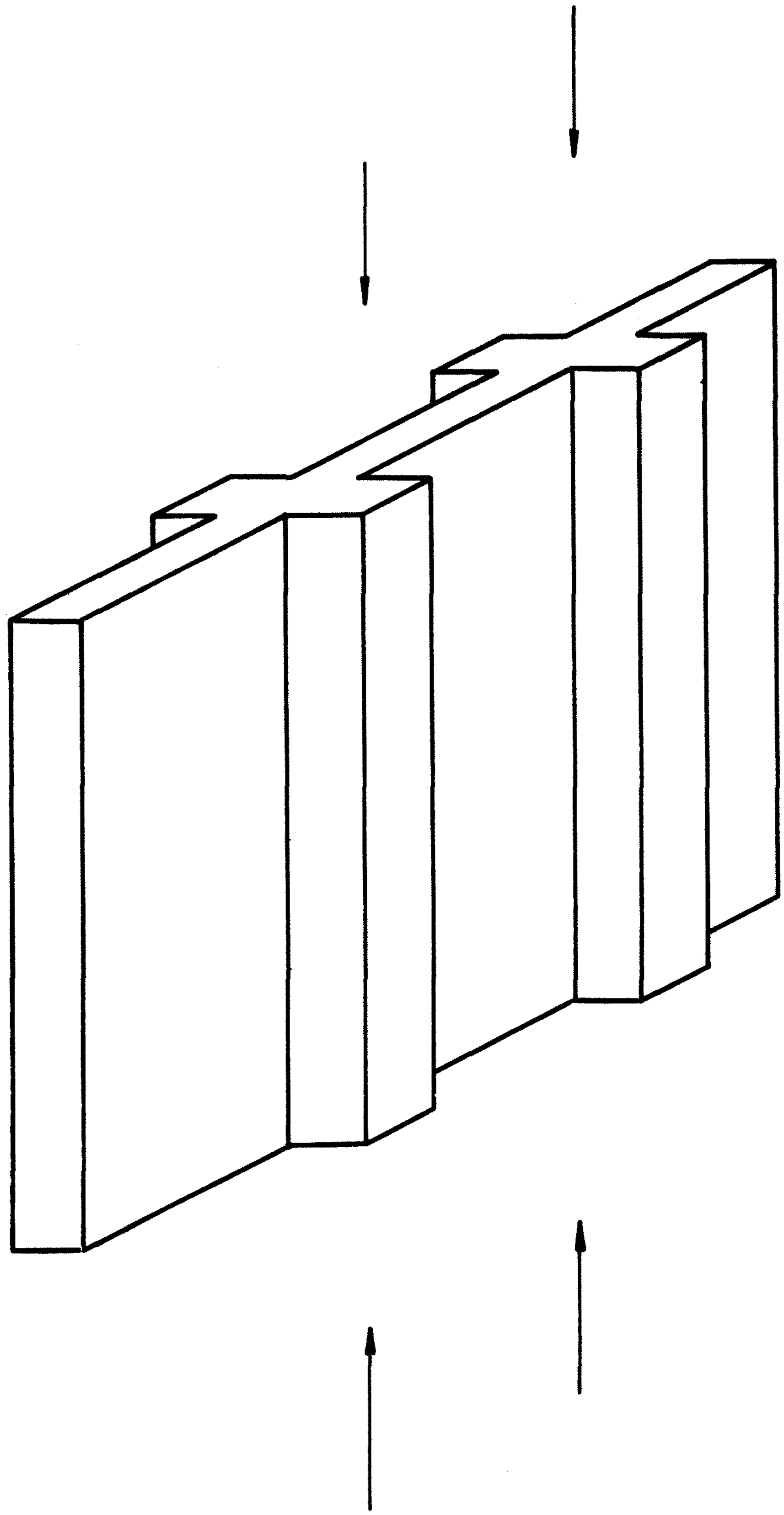


Fig. 1 Integral reinforced flat panel

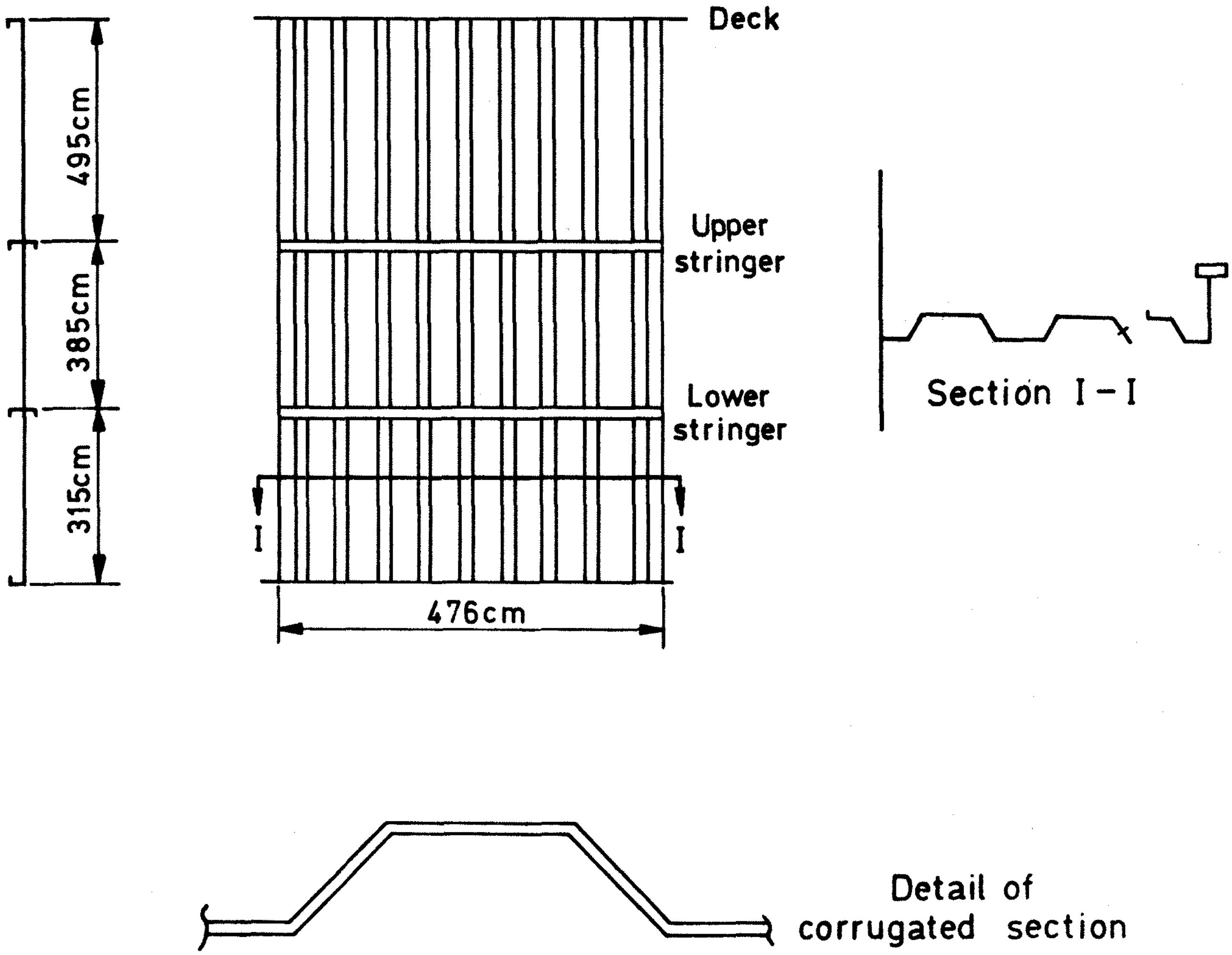


Fig. 2 Ship bulkhead problem

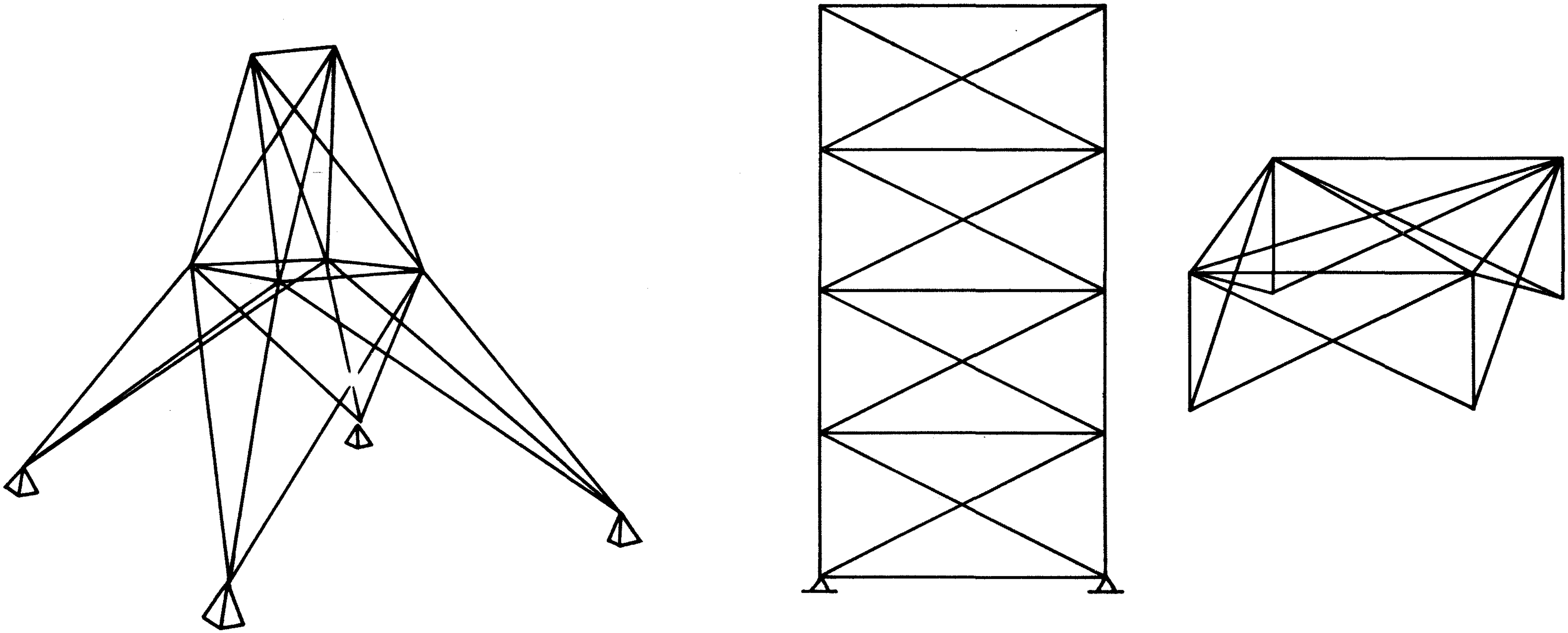


Fig. 3 Typical examples of pin-jointed frame problems solved by reduced geometric programming

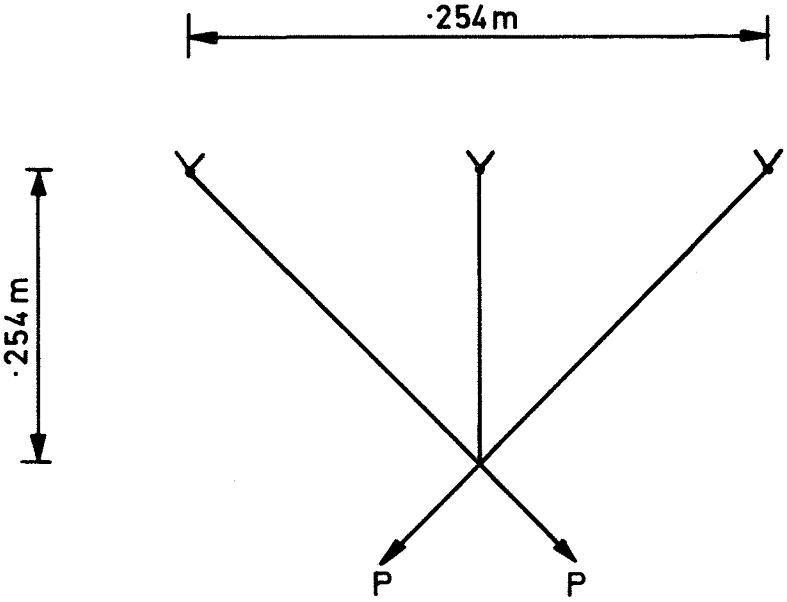


Fig. 4a Multiple loaded three bar truss

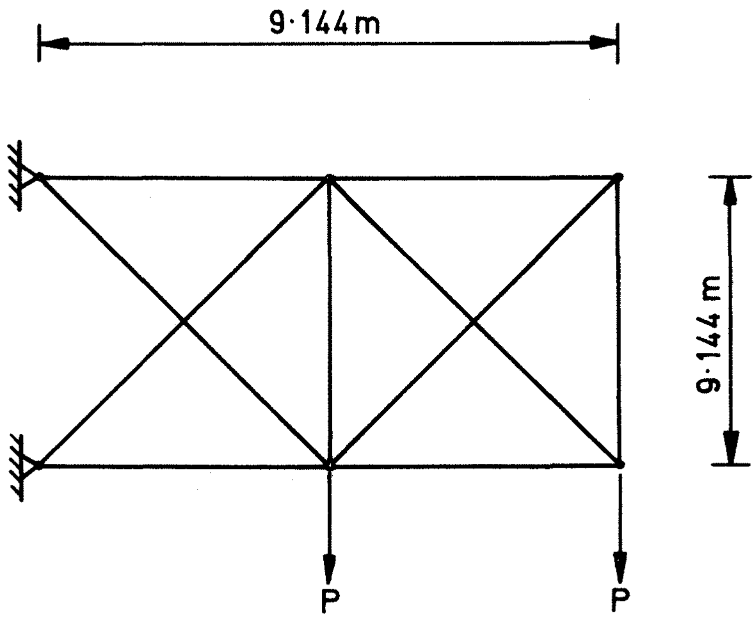


Fig. 4b Single loaded ten bar framework

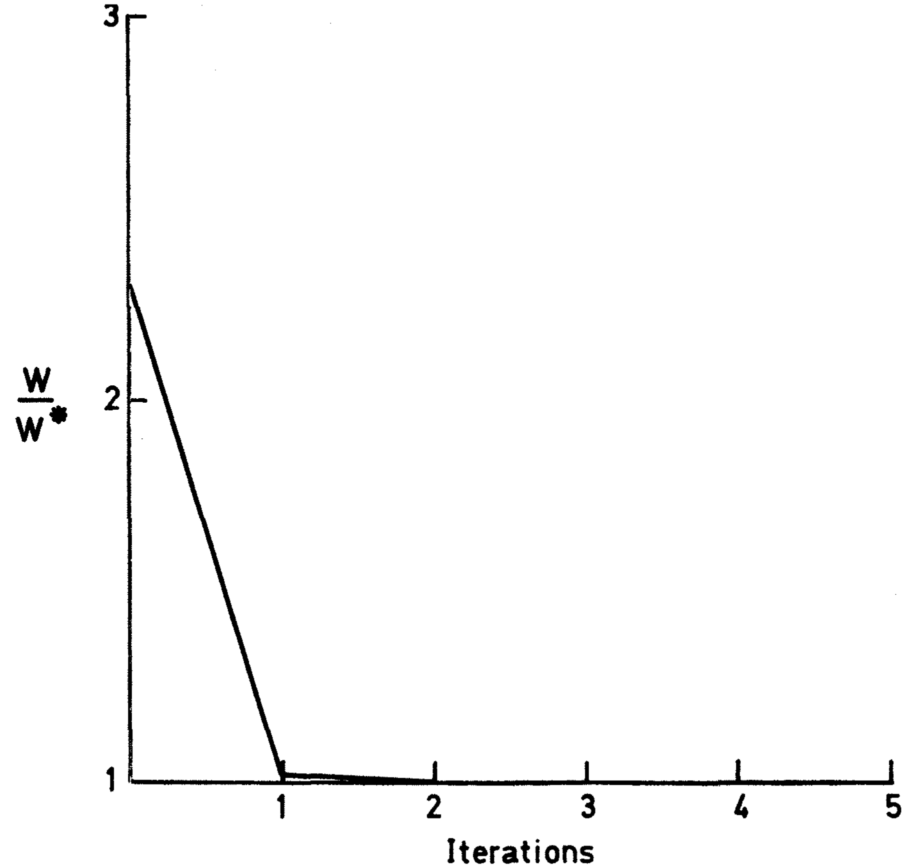


Fig. 5a Results for three bar truss

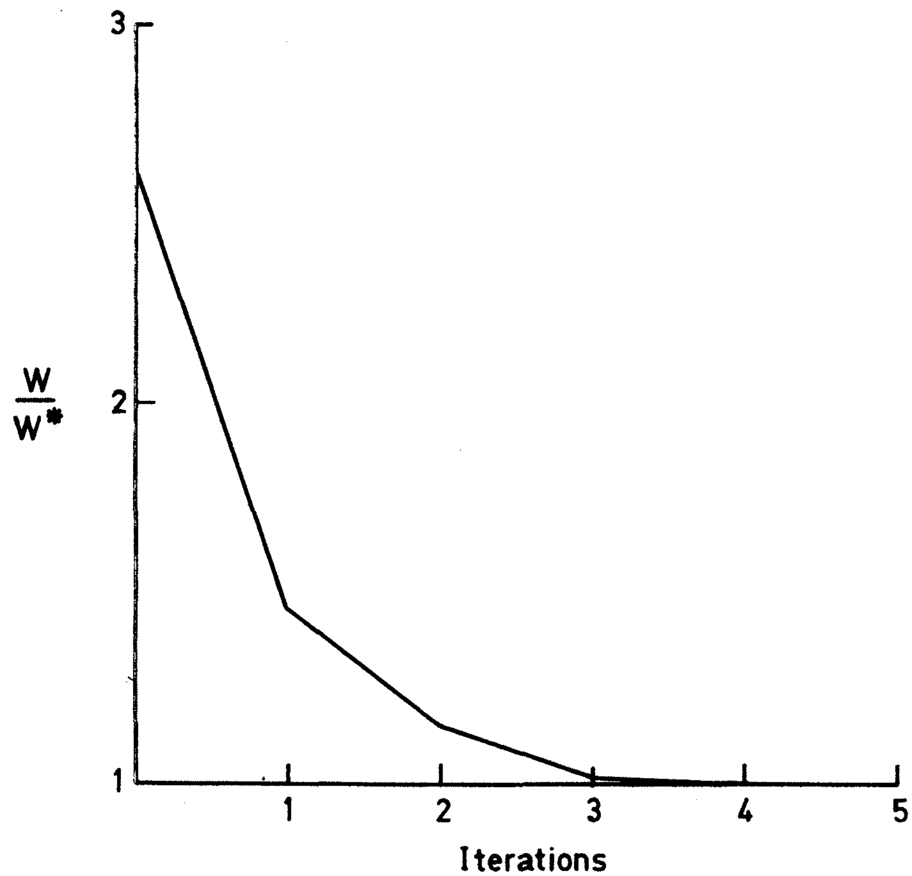
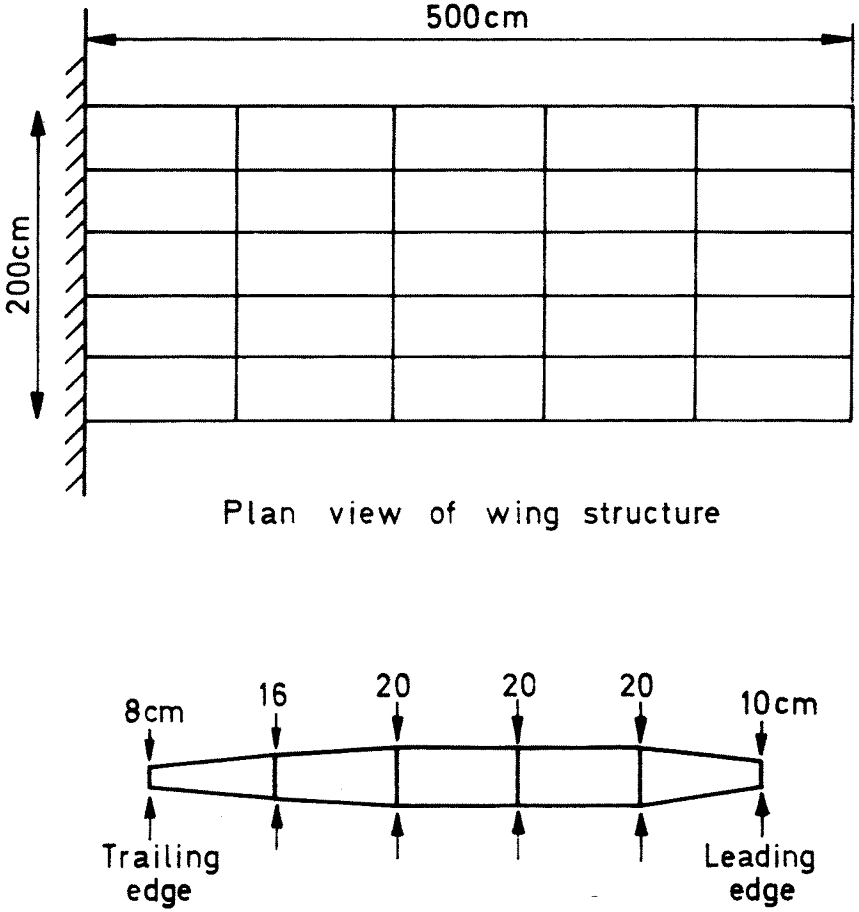


Fig. 5b Results for ten bar framework



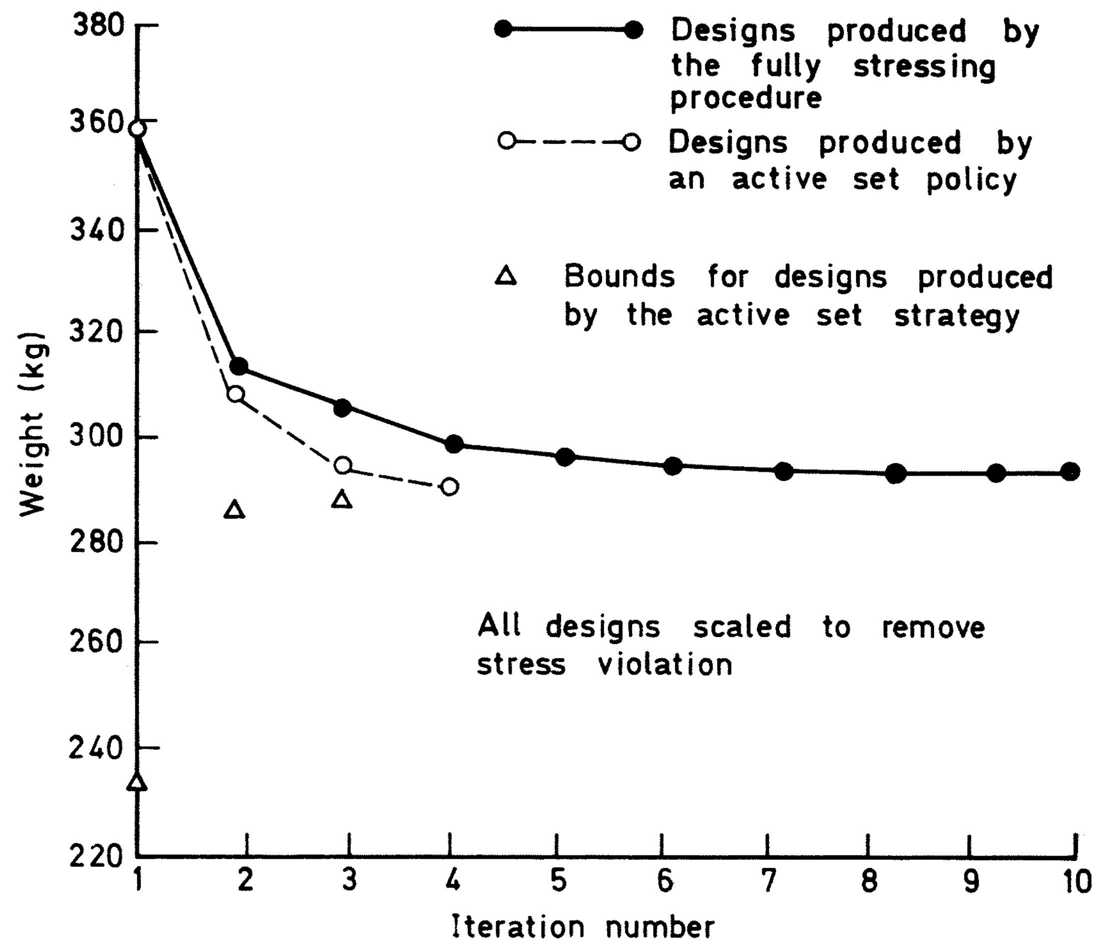


Fig. 6 Stress constrained wing box